On Quasiconvexity and Relatively Hyperbolic Structures on Groups

Eduardo Martínez-Pedroza
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario L8S 4K1. Canada
emartinez@math.mcmaster.ca

Abstract

Let G be a group which is hyperbolic relative to a collection of subgroups $\mathcal{H}_1$, and it is also hyperbolic relative to a collection of subgroups $\mathcal{H}_2$ such that $\mathcal{H}_2 \subset \mathcal{H}_1$. In this note we characterize when a subgroup Q of G which is quasiconvex with respect to $\mathcal{H}_1$ is also quasiconvex with respect to H_2. We also show that if a subgroup Q of G is quasiconvex with respect $\mathcal{H}_2$, then it is quasiconvex with respect to $\mathcal{H}_1$. Some applications are discussed.

1 Introduction

If G is a countable group and $\mathcal{H}$ is a collection of subgroups of G, the notion of relative hyperbolicity for the pair $(G, \mathcal{H})$ has been defined by different authors [4, 6, 7, 10, 11, 12, 17, 19]. All these definitions are equivalent when the group G is finitely generated [6, 10, 17, 18, 19]. When a pair $(G, \mathcal{H})$ satisfies the relative hyperbolicity condition we say that the group G is hyperbolic relative to the peripheral structure $\mathcal{H}$, and when the collection $\mathcal{H}$ is fixed we just say that the group G is relatively hyperbolic.

For a group G hyperbolic relative to a collection of subgroups $\mathcal{H}$, Dahmani [5] and Osin [17] studied classes of subgroups of G which they called quasiconvex relative to $\mathcal{H}$, intending to generalize the notion of quasiconvexity in word hyperbolic groups. Hruska introduced several notions of relative quasiconvexity in the setting of countable (not necessarily finitely generated) relatively hyperbolic groups, including the notions based on Osin's and Dahmani's, and showed that they are equivalent [12].

A group G may be relatively hyperbolic with respect to different peripheral structures, see for example [16]. Suppose that $\mathcal{Q}$ is the class of quasiconvex subgroups of G relative to the peripheral structure $\mathcal{H}$. The goal of this paper is to study how $\mathcal{Q}$ varies when $\mathcal{H}$ changes.

If G is a group generated by X and $g \in G$, we will denote by $|g|_X$ the distance from the identity element to g in the word metric induced by X. For a subgroup

$Q < G$ and an element $g \in G$, we will denote by Q^g the subgroup $g^{-1}Qg$. The main result of the paper is the following.

Theorem 1.1. *Let G be a group with a finite generating set X. Suppose that G is hyperbolic relative to a collection of subgroups $\mathcal{H}_1$, G is also hyperbolic relative to a collection of subgroups $\mathcal{H}_2$, and $\mathcal{H}_2 \subset \mathcal{H}_1$.*

1. *If R is a quasiconvex subgroup of G relative to $\mathcal{H}_2$, then R is quasiconvex relative to $\mathcal{H}_1$.*
2. *Suppose that R is a quasiconvex subgroup of G relative to $\mathcal{H}_1$. If for any $Q \in \mathcal{H}_1 \setminus \mathcal{H}_2$ and any $g \in G$ the subgroup $R \cap gQg^{-1}$ is quasiconvex relative to $\mathcal{H}_2$, then R is quasiconvex relative to $\mathcal{H}_2$.*

Recall that a group is *elementary* if it contains an cyclic group of finite index. Since cyclic subgroups are quasiconvex [17, Corollary 4.20], elementary subgroups are also quasiconvex.

Corollary 1.2. *Let G be a group which is hyperbolic relative to the collection of subgroups $\mathcal{H}$ and $\mathcal{H}'$. Suppose that $\mathcal{H} \subset \mathcal{H}'$ and every $H \in \mathcal{H}' \setminus \mathcal{H}$ is elementary.*

Then a subgroup of G is quasiconvex relative to $\mathcal{H}$ if and only if it is quasiconvex relative to $\mathcal{H}'$.

Proof. The "only if" part is the first part of Theorem 1.1. Suppose $R < G$ is quasiconvex relative to $\mathcal{H}'$. For every $H \in \mathcal{H}' \setminus \mathcal{H}$ and $g \in G$, the subgroup $R \cap H^g$ is an elementary subgroup, in particular quasiconvex relative to $\mathcal{H}$. By the second part of Theorem 1.1, R is quasiconvex relative to $\mathcal{H}$. □

The following result is a Corollary of Theorem 1.1 and previous work of the author [15, Theorem 1.1]. It is a combination theorem of quasiconvex subgroups and hyperbolically embedded subgroups which was proved for the case of word-hyperbolic groups by R. Gitik [8, Theorem 2].

Corollary 1.3. *Let G be a hyperbolic group relative to a collection of subgroups $\mathcal{H}$, and suppose that X is a finite generating set of G.*

For every relatively quasiconvex subgroup Q, and every hyperbolically embedded subgroup P, there is constant $C = C(Q, P, X) \geq 0$ with the following property. If R is a subgroup of P such that

1. *$Q \cap P \leq R$, and*
2. *$|g|_X \geq C$ for any $g \in R \setminus Q$,*

then the natural homomorphism

$$Q *_{Q \cap R} R \longrightarrow G$$

is injective.
Moreover, if the subgroup R is quasiconvex relative to $\mathcal{H}$, then the subgroup $\langle Q \cup R \rangle$ is quasiconvex relative to $\mathcal{H}$.

An interesting combination theorem for cyclic subgroups of relatively hyperbolic groups was proved by G. Arzhntseva and A. Minasyan [2, Theorem 1.1]. They used this combination theorem to prove that relatively hyperbolic groups with no non-trivial finite normal subgroups are C^*-simple. Corollary 1.3 allows us to obtain a slight refinement of their combination result.

Corollary 1.4. *Let G be a non-elementary and properly relatively hyperbolic group with respect to a collection of subgroups $\mathcal{H}$. Suppose that G has no non-trivial finite normal subgroups. Then for any finite subset F of non-trivial elements of G there exists an element $g \in G$ with the following properties. For every $f \in F$,*

1. *$\langle f, g \rangle$ is isomorphic to the free product $\langle f \rangle * \langle g \rangle$, and*
2. *$\langle f, g \rangle$ is a quasiconvex subgroup relative to $\mathcal{H}$.*

Remark 1.5. The original result by G. Arzhntseva and A. Minasyan [2, Theorem 1.1] does not include the statement on quasiconvexity of the subgroups $\langle f, g \rangle$.

Suppose that G is a word-hyperbolic group. As described in [1, Section 3], a quasiconvex subgroup $H < G$ induces a peripheral structure $\mathcal{P}$ of G. The construction of the peripheral structure $\mathcal{P}$ of G induced by H is recalled in Section 4.4 below. As a Corollary of Theorem 1.1 we recover the following result from [1] without the assumption that G is torsion free.

Corollary 1.6. *[1, Proposition 3.12] Let H be a quasiconvex subgroup of word–hyperbolic group G. Let $\mathcal{P}$ be the peripheral structure on G induced by H. Then*

1. *G is hyperbolic relative to $\mathcal{P}$, and*
2. *H is a relatively quasiconvex subgroup of G with respect to the peripheral structure $\mathcal{P}$.*

Remark 1.7. The definition of relative quasiconvexity in the paper [1] differs from definition 2.10 of relative quasiconvexity in this paper. The equivalence of these two definitions is a result by Manning and the author [14, Theorem A.100]. For more on different definitions of relative quasiconvexity see remark 2.11 below.

The paper is organized as follows. Section 2 introduces background and notation. In section 3 a technical result on the geometry of hyperbolically embedded subgroups is proved. In the last section the proofs of the main result and corollaries are discussed.

Acknowledgements: The author thanks Noel Brady, Jason Manning, and Ashot Minasyan for useful comments. This project was partially supported by NSF grant no. DMS-0505707 and a Britton Posdoctoral Fellowship at McMaster University.

2 Definitions and Background

2.1 Relative hyperbolicity

The notion of relative hyperbolicity has been studied by several authors with different equivalent definitions. The definition in this subsection is based on the

work by Osin in [17]. Let G be a group, $\mathcal{H}$ denote a collection of subgroups $\{H_1, \ldots, H_m\}$, and X be a finite generating set which is assumed to be symmetric, i.e, $X = X^{-1}$. Denote by $\Gamma(G, \mathcal{H}, X)$ the Cayley graph of G with respect to the generating set $X \cup \bigcup \mathcal{H}$. If p is a path between vertices in $\Gamma(G, \mathcal{H}, X)$, we will refer to its initial vertex as p_-, and its terminal vertex as p_+. The path p determines a word $Label(p)$ in the alphabet $X \cup \bigcup \mathcal{H}$ which represents an element g so that $p_+ = p_- g$. The length of a path p will be denoted by $l(p)$.

Definition 2.1 (Weak Relative Hyperbolicity)**.** The pair $(G, \mathcal{H})$ is *weakly relatively hyperbolic* if there is an integer $\delta \geq 0$ such that $\Gamma(G, \mathcal{H}, X)$ is δ–hyperbolic. We may also say that G is *weakly relatively hyperbolic, relative to* $\mathcal{H}$.

Definition 2.2 ([17])**.** Let q be a combinatorial path in the Cayley graph $\Gamma(G, \mathcal{H}, X)$. Sub-paths of q with at least one edge are called *non-trivial.* For $H_i \in \mathcal{H}$, an H_i–*component* of q is a maximal non-trivial sub-path s of q with $Label(s)$ a word in the alphabet H_i. When we don't need to specify the index i, we will refer to H_i–components as $\mathcal{H}$*–components.*

Two $\mathcal{H}$–components s_1, s_2 are *connected* if the vertices of s_1 and s_2 belong to the same left coset of H_i for some i. A $\mathcal{H}$–component s of q is *isolated* if it is not connected to a different $\mathcal{H}$–component of q. The path q is *without backtracking* if every $\mathcal{H}$–component of q is isolated.

A vertex v of q is called *phase* if it is not an interior vertex of a $\mathcal{H}$–component s of q. Let p and q be paths between vertices in $\Gamma(G, \mathcal{H}, X)$. The paths p and q are k*–similar* if

$$\max\{dist_X(p_-, q_-), dist_X(p_+, q_+)\} \leq k,$$

where $dist_X$ is the metric induced by the finite generating set X (as opposed to the metric in $\Gamma(G, \mathcal{H}, X)$).

Remark 2.3. A geodesic path q in $\Gamma(G, \mathcal{H}, X)$ is without backtracking, all $\mathcal{H}$–components of q consist of a single edge, and all vertices of q are phase.

Definition 2.4 (Bounded Coset Penetration (BCP))**.** The pair $(G, \mathcal{H})$ satisfies the *BCP property* if for any $\lambda \geq 1$, $c \geq 0$, $k \geq 0$, there exists an integer $\epsilon(\lambda, c, k) > 0$ such that for p and q any two k–similar (λ, c)–quasi-geodesics in $\Gamma(G, \mathcal{H}, X)$ without backtracking, the following conditions hold:

(i.) The sets of phase vertices of p and q are contained in the closed $\epsilon(\lambda, c, k)$–neighborhoods of each other, with respect to the metric $dist_X$.

(ii.) If s is any $\mathcal{H}$–component of p such that $dist_X(s_-, s_+) > \epsilon(\lambda, c, k)$, then there exists a $\mathcal{H}$–component t of q which is connected to s.

(iii.) If s and t are connected $\mathcal{H}$–components of p and q respectively, then

$$\max\{dist_X(s_-, t_-), dist_X(s_+, t_+)\} \leq \epsilon(\lambda, c, k).$$

Remark 2.5. Our definition of the BCP property corresponds to the conclusion of Theorem 3.23 in [17].

Definition 2.6 (Relative Hyperbolicity)**.** The pair $(G, \mathcal{H})$ is *relatively hyperbolic* if the group G is weakly relatively hyperbolic relative to $\mathcal{H}$ and the pair $(G, \mathcal{H})$ satisfies the Bounded Coset Penetration property. If $(G, \mathcal{H})$ is relatively hyperbolic then we say G is *relatively hyperbolic, relative to* $\mathcal{H}$; if there is no ambiguity, we just say that G is relatively hyperbolic.

Remark 2.7. Definition 2.6 given here is equivalent to Osin's [17, Definition 2.35] for finitely generated groups: To see that Osin's definition implies 2.6, apply [17, Theorems 3.23]; to see that 2.6 implies Osin's definition, apply [17, Lemma 7.9 and Theorem 7.10]. For the equivalence of Osin's definition and the various other definitions of relative hyperbolicity see [12] and the references therein.

The following corollary is a straight forward application of the BCP-property.

Corollary 2.8. *[15, Corollary 2.8] Let G be a hyperbolic group relative to a collection of subgroups $\mathcal{H}$, and X is a finite generating set of G. Let g_1H_i and g_2H_j be different left cosets. For any pair of geodesics p and q in $\Gamma(G, \mathcal{H}, X)$ such that $p_-, q_- \in g_1H_i$, $p_+, q_+ \in g_2H_j$, and neither p nor q have more than one vertex in g_1H_i or g_2H_j, the following holds.*

1. *$l(q) \leq l(p) + 2$, and*

2. *q and p are $\epsilon(1, 4, 0)$-similar.*

2.2 Parabolic and Relatively Quasiconvex Subgroups

Let G be a group, $\mathcal{H} = \{H_i\}_{i=1}^m$ be a collection of subgroups of G, and X be a finite generating set for G. Suppose that G is hyperbolic relative to $\mathcal{H}$.

Definition 2.9. The *peripheral* subgroups of G are the elements of $\mathcal{H}$. A subgroup of G is called *parabolic* if it can be conjugated into a peripheral subgroup.

Definition 2.10. [17, Definition 4.9] A subgroup Q of G is called *quasiconvex relative to* $\mathcal{H}$ (or simply *relatively quasiconvex* when the collection $\mathcal{H}$ is fixed) if there exists a constant $\sigma \geq 0$ such that the following holds: Let f, g be two elements of Q, and p an arbitrary geodesic path from f to g in the Cayley graph $\Gamma(G, \mathcal{H}, X)$. For any vertex $v \in p$, there exists a vertex $w \in Q$ such that $dist_X(v, w) \leq \sigma$, where $dist_X$ is the word metric induced by X.

Remark 2.11. Dahmani [5] and Osin [17] studied classes of subgroups of relatively hyperbolic groups which they called relatively quasiconvex, intending to generalize the notion of quasiconvexity in hyperbolic groups. Hruska introduced several notions of relative quasiconvexity in the setting of countable (not necessarily finitely generated) relatively hyperbolic groups, including the notions based on Osin's and Dahmani's, and showed that they are equivalent [12].

Remark 2.12. The BCP-property implies that any parabolic subgroup is quasiconvex.

Proposition 2.13. *[12, Corollary 9.5] [15, Proposition 1.3] Let Q and R be relatively quasiconvex subgroups of G. Then $Q \cap R$ is a relatively quasiconvex subgroup of G.*

Proposition 2.14. *[12, Theorem 9.1] [15, Proposition 1.5] Let Q be a σ-quasiconvex subgroup of G. Then any infinite maximal parabolic subgroup of Q is conjugate by an element of Q to a subgroup in the set*

$$\{Q \cap H^z : H \in \mathcal{H} \text{ and } z \in G \text{ with } |z|_X \leq \sigma\}.$$

In particular, the number of infinite maximal parabolic subgroups up to conjugacy in Q is finite.

Theorem 2.15. *[16, Theorem 1.5] Let P be a hyperbolically embedded subgroup of G relative to $\mathcal{H}$. Then P is a quasiconvex subgroup relative to $\mathcal{H}$.*

2.3 On the intersection of subgroups in countable groups.

A version of the following result can be found in [15], and for the interested reader a more general version appears in [12]. It will be used in the proof of Theorem 1.1.

Proposition 2.16. *[12, Proposition 9.4] [15, Lemma 4.2] Let A be a countable group with a proper left invariant metric d. Then for any B and C subgroups of A, and any constant $K \geq 0$, there exists $M = M(B, C, K) \geq 0$ so that*

$$B \cap N_K(gC) \subset N_M(B \cap C^g),$$

where $N_K(gC)$ and $N_M(B \cap C^g)$ denote the closed K-neighborhood and the closed M-neighborhood of gC and $B \cap C^g$ in (A, d) respectively.

3 Quasigeodesics and Hyperbolically embedded subgroups

Proposition 3.6 in this section is the main technical tool in the proof of the main theorem. Suppose that G is a hyperbolic group relative to $\mathcal{H}$, and $Q < G$ is a subgroup such that G also relative to $\mathcal{H} \cup \{Q\}$. A subgroup $Q < G$ with this property is called hyperbolically embedded. Proposition 3.6 shows that certain paths in $\Gamma(G, \mathcal{H}, X)$ induced by geodesics in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ are quasigeodesics.

The section consists of three parts. First, a result from [17] is discussed. In the second subsection, some results on hyperbolically embedded subgroups from [16] are recalled. This subsection contains a proof Corollary 3.5 which is a result that should be known by most experts in the field but there was no reference available. The last subsection consists of the statement and proof of Proposition 3.6.

3.1 A Result about Polygons by Osin

The following proposition is a direct consequence of [17, Lemma 2.27].

Proposition 3.1. *[17] Let G be a group generated by a finite set X and hyperbolic relative to a collection of subgroups $\mathcal{H}$.*

There exists a finite subset $\Omega \subset G$ and a constant $D > 0$ satisfying the following conditions.

1. *Every $H \in \mathcal{H}$ is generated by $\Omega_H = \Omega \cap H$.*

2. *Let q be a cycle in $\Gamma(G, \mathcal{H}, X)$, $H \in \mathcal{H}$, and $S = \{p_1, \ldots, p_k\}$ a certain set of isolated H-componets of q. If the length of q is n, then*

$$\sum_{p \in S} dist_{\Omega_H}(p_-, p_+) \leq D\ n.$$

In particular,

$$\sum_{p \in S} dist_X(p_-, p_+) \leq M\ D\ n,$$

where $M = \max_{\omega \in \Omega}\{|\omega|_X\}$.

Proof. The set Ω is defined in [17, Definition 2.25]. The first statement of the proposition corresponds to [17, Proposition 2.29]. The relative area of the cycle q in $\Gamma(G, \mathcal{H}, X)$ is a natural number denoted by $Area^{rel}(q)$, see [17, Definition 2.26]. Then [17, Lemma 2.27] states that $\sum_{p \in S} dist_{\Omega_H}(p_-, p_+) \leq K\ Area^{rel}(q)$ where K is a constant independent of q and the collection of H-components S. Since G is hyperbolic relative to $\mathcal{H}$, then $Area^{rel}(q) \leq Ln$ where L is a constant independent of q and S (See [17, Definition 2.35] and remark 2.7). To finish the proof let $D = KL$. $\square$

3.2 Hyperbolically Embedded Subgroups

Let G be a group, $\mathcal{H} = \{H_i\}_{i=1}^m$ be a collection of subgroups of G, and X be a finite generating set for G. Suppose that G is hyperbolic relative to $\mathcal{H}$.

Definition 3.2. [16, Definition 1.4] A subgroup $Q \leq G$ is said to be *hyperbolically embedded into G relative to $\mathcal{H}$*, if G is hyperbolic relative to $\mathcal{H} \cup \{Q\}$.

The hyperbolically embedded groups where characterized by Bowditch [4, Theorem 7.11] for the case that G is a word hyperbolic group, and by Osin [16, Theorem 1.5] for the general case.

Theorem 3.3. *[16, Theorem 1.5] A subgroup Q of G is hyperbolically embedded relative to $\mathcal{H}$, if the following conditions hold.*

1. *Q is generated by a finite set Y.*

2. *There exists $\lambda, c \geq 0$ such that for any element $g \in Q$, we have $|g|_Y \leq \lambda dist_{X \cup \mathcal{H}}(1, g) + c$.*

3. *For any $g \in G$ such that $g \notin Q$, the intersection $Q \cap Q^g$ is finite.*

In particular, the first and second statement of the characterization of hyperbolically embedded subgroups implies relative quasiconvexity.

Corollary 3.4. *A subgroup $Q < G$ which is hyperbolically embedded relative to $\mathcal{H}$ is quasiconvex relative to $\mathcal{H}$.*

Corollary 3.5. *Let G be a group with a finite generating set X. Suppose that G is hyperbolic relative to a collection of subgroups $\mathcal{H}_1$, G is also hyperbolic relative to a collection of subgroups $\mathcal{H}_2$, and $\mathcal{H}_2 \subset \mathcal{H}_1$.*

Then for any collection of subgroups $\mathcal{H}$ such that $\mathcal{H}_2 \subset \mathcal{H} \subset H_1$, the group G is hyperbolic relative to $\mathcal{H}$. In particular, every element of $\mathcal{H}_1 \setminus \mathcal{H}_2$ is a hyperbolically embedded subgroup of G relative to $\mathcal{H}_2$.

Proof. Suppose that $\mathcal{H} = \mathcal{H}_2 \cup \{Q\}$. The general case follows by induction with this assumption as the base and induction case. It is enough to show that Q is hyperbolically embedded relative to $\mathcal{H}_2$. Conditions (1) and (3) of Theorem 3.3 are satisfied by Q since $\mathcal{H}_1$ is a peripheral structure for G, see for example [17, Theorem 1.1 and Proposition 2.36]. To show that S satisfies condition (2) of Theorem 3.3 we will use Proposition 3.1.

Let $D > 0$ and $\Omega \subset G$ be given by Proposition 3.1 applied to G as a hyperbolic group relative to $\mathcal{H}_1$. Then $Y = \Omega \cap Q$ is a finite generating set for Q. Regard $\Gamma(G, \mathcal{H}_2, X)$ as a subgraph of $\Gamma(G, \mathcal{H}_1, X)$ in the obvious way.

Let g be an element of Q. Let r be a geodesic in $\Gamma(G, \mathcal{H}_2, X)$ from the identity element to g. Since Q is an element of $\mathcal{H}_1$, there is an edge p in $\Gamma(G, \mathcal{H}_1, X)$ from the identity element to g. Consider r as a polygonal path, and let $\mathcal{P}$ be the closed polygon formed by r and p in $\Gamma(G, \mathcal{H}_1, X)$. Since r has no Q-components, p is an isolated Q-component of $\mathcal{P}$. Since the length of the closed path $\mathcal{P}$ is $(dist_{X \cup \mathcal{H}_2}(1, g) + 1)$, Theorem 3.1 implies

$$|g|_Y = dist_Y(p_-, p_+) \leq D \; dist_{X \cup \mathcal{H}_2}(1, g) + D.$$

Therefore Q satisfies condition (2) of Theorem 3.3 and therefore Q is hyperbolically embedded relative to $\mathcal{H}_2$. ☐

3.3 Technical Result on Quasigeodesics and Hyperbolically Embedded Groups

Let G be a group generated by a finite set X and hyperbolic relative to a collection of subgroups $\mathcal{H}$. Suppose that Q is a hyperbolically embedded subgroup of G. Let $dist_{X \cup \mathcal{H}}$ and $dist_{X \cup \mathcal{H} \cup \{Q\}}$ denote the metrics in $\Gamma(G, \mathcal{H}, X)$ and $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ respectively.

Proposition 3.6. *There exist constants $\lambda = \lambda(G, X, \mathcal{H}, Q) \geq 1$ and $c = c(G, X, \mathcal{H}, Q) \geq 0$ with the following property.*

If p is a geodesic in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ and q is a path in $\Gamma(G, \mathcal{H}, X)$ obtained by replacing each Q-component of p by a geodesic segment in $\Gamma(G, \mathcal{H}, X)$ connecting its endpoints, then q is a (λ, c)-quasi-geodesic in $\Gamma(G, \mathcal{H}, X)$.

The proof of Proposition 3.6 is based in Lemma 3.7 whose proof is simitar to the proof of Corollary 3.5.

Lemma 3.7. *For any $\lambda \geq 1$ and $c \geq 0$, there exist a constant*

$$L = L(G, X, \mathcal{H}, Q, \lambda, c)$$

with the following property.

Let p be any path in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ such that

1. *p is without backtracking,*
2. *p is of the form*

$$p = s_1 t_1 \ldots s_k t_k s_{k+1},$$

where each s_i is a geodesic segment in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$, and $\{t_i\}_{i=1}^k$ are all the Q-components of p.

3. *p is a (λ, c)-quasi-geodesic.*

Then the path q in $\Gamma(G, \mathcal{H}, X)$

$$q = s_1 u_1 \ldots s_k u_k s_{k+1},$$

where each u_i is a geodesic segment in $\Gamma(G, \mathcal{H}, X)$ connecting the endpoints of t_i, satisfies

$$l(q) \leq L \; dist_{X \cup \mathcal{H}}(q_-, q_+) + L.$$

Proof. Since Q is hyperbolically embedded, the group G is hyperbolic relative to $\mathcal{H} \cup \{Q\}$. Let $D > 0$ and $\Omega \subset G$ be given by Proposition 3.1 applied to G as a hyperbolic group relative to $\mathcal{H} \cup \{Q\}$. Regard $\Gamma(G, \mathcal{H}, X)$ as a subgraph of $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ in the obvious way. Let r be a geodesic in $\Gamma(G, \mathcal{H}, X)$ connecting the endpoints of p. In $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$, consider r as a polygonal path and let $\mathcal{P}$ be the closed polygon in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ given by

$$\mathcal{P} = s_1 t_1 \ldots s_k t_k s_{k+1} r.$$

Since p is a $(\lambda, c) - -$quasigeodesic, the length of $\mathcal{P}$ satisfies

$$\begin{aligned} l((P)) &\leq l(p) + l(r) \\ &\leq (\lambda + 1) dist_{X \cup \mathcal{H}}(p_-, p_+) + c. \end{aligned}$$

Since each t_i is an isolated Q-component of $\mathcal{P}$, Proposition 3.1 and the above inequality implies

$$\begin{aligned} \sum_{i=1}^{k} l(u_i) &\leq \sum_{i=1}^{k} dist_X((t_i)_-, (t_i)_+) \\ &\leq D \; l(\mathcal{P}) \\ &\leq (\lambda + 1) D \; dist_{X \cup \mathcal{H}}(p_-, p_+) + c \; D. \end{aligned}$$

Since p is (λ, c)–quasigeodesic, we have

$$\begin{aligned} l(q) &= \sum_{i=1}^{k+1} l(s_i) + \sum_{i=1}^{k} l(u_i) \\ &= l(p) + \sum_{i=1}^{k} l(u_i) \\ &\leq (\lambda D + D + \lambda)\ dist_{X \cup \mathcal{H}}(p_-, p_+) + c\ D + c, \end{aligned}$$

which finish the proof of the Lemma. □

Proof of Proposition 3.6. Decompose the path p as

$$p = s_1 t_1 \ldots s_k t_k s_{k+1},$$

where $\{t_i\}_{i=1}^{k}$ are all the Q-components of p. Then the path q in $\Gamma(G, \mathcal{H}, X)$ decomposes as

$$q = s_1 u_1 \ldots s_k u_k s_{k+1},$$

where each u_i is a geodesic segment in $\Gamma(G, \mathcal{H}, X)$ connecting the endpoints of t_i.

Let q' be a subpath of q. We consider two cases:

Case 1. Suppose that q' is of the form

$$q' = s'_\imath u_\imath \ldots s_{\imath+\jmath} u_{\imath+\jmath} s'_{\imath+\jmath+1},$$

where $s'_\imath$ and $s'_{\imath+\jmath+1}$ are subpaths of $s_\imath$ and $s_{\imath+\jmath+1}$ respectively. In this case, Lemma 3.7 implies that

$$l(q') \leq L_1\ dist_{X \cup \mathcal{H}}(q'_-, q'_+) + L_1,$$

where $L_1 = L(G, X, \mathcal{H}, Q, 1, 0)$.

Case 2. Otherwise, at least one of the endpoints of q' is a vertex of u_i for some i. There are three similar cases to consider. We only consider one and leave the other two for the reader. Suppose that q' is of the form

$$q' = u'_\imath s_{\imath+1} u_{\imath+1} \ldots u_{\imath+\jmath-1} s_{\imath+\jmath} u'_{\imath+\jmath},$$

where $u'_\imath$ and $u'_{\imath+\jmath}$ are subpaths of $u_\imath$ and $u_{\imath+\jmath}$ respectively. Let p' be the path in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ given by

$$p' = t'_\imath s_{\imath+1} t_{\imath+1} \ldots t_{\imath+\jmath-1} s_{\imath+\jmath} t'_{\imath+\jmath},$$

where $t'_\imath$ and $t'_{\imath+\jmath}$ correspond to single edges connecting the endpoints of $u'_\imath$ and $u'_{\imath+\jmath}$ respectively. Corollary 2.8 shows that p' is a $(4, 0)$-quasi-geodesic in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$. Lemma 3.7 implies that

$$l(q') \leq L_2\ dist_{X \cup \mathcal{H}}(q'_-, q'_+) + L_2,$$

where $L_2 = L(G, X, \mathcal{H}, Q, 4, 0)$. To finish the proof define $\lambda = c = max\{L_1, L_2\}$. □

4 Proofs of the Main Results

4.1 Proof of Theorem 1.1

Proposition 4.1. *Suppose that G is hyperbolic group relative to a collection of subgroups $\mathcal{H}$ with finite generating set X, and Q is a hyperbolically embedded subgroup of G.*

If R is a quasiconvex subgroup of G relative to $\mathcal{H}$, then R is quasiconvex relative to $\mathcal{H} \cup \{Q\}$.

Proof. Let f be an element of R and let p be a geodesic in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ from 1 to f. Let q be the path in $\Gamma(G, \mathcal{H}, X)$ obtained by replacing each Q-component of p by geodesic segments in $\Gamma(G, \mathcal{H}, X)$. Proposition 3.6 implies that q is a (λ, c)-quasi-geodesic in $\Gamma(G, \mathcal{H}, X)$, where the constants λ and c are independent of the element f and the path p. Since R is σ-quasiconvex relative to $\mathcal{H}$, the BCP property implies that for any vertex u of q (in particular, for any vertex of p) there is a vertex $v \in R$ such that $dist_X(u, v) \leq \epsilon(\lambda, c, 0) + \sigma$. It follows that R is $(\epsilon(\lambda, c, 0) + \sigma)$-quasiconvex relative to $\mathcal{H} \cup \{Q\}$. □

Proposition 4.2. *Suppose that G is hyperbolic group relative to a collection of subgroups $\mathcal{H}$ with finite generating set X, and Q is a hyperbolically embedded subgroup of G.*

If R is a σ-quasiconvex subgroup of G relative to $\mathcal{H} \cup \{Q\}$ and for any $g \in G$ with $|g|_X \leq \sigma$ the subgroup $R \cap gQg^{-1}$ is quasiconvex relative to $\mathcal{H}$, then R is quasiconvex relative to $\mathcal{H}$.

Proof. Let f be an element of R. Let q be a geodesic in $\Gamma(G, \mathcal{H} \cup \{Q\}, X)$ from 1 to f and p be the path in $\Gamma(G, \mathcal{H}, X)$ obtained by replacing each Q-component of q by geodesic segments in $\Gamma(G, \mathcal{H}, X)$. Proposition 3.6 implies that p is a (λ, c)-quasi-geodesic in $\Gamma(G, \mathcal{H}, X)$ with λ and c independent of f and q. Decompose the path q as

$$q = s_1 t_1 \dots s_k t_k s_{k+1},$$

where $\{t_i\}_{i=1}^k$ are all the Q-components of q. Then the path p in $\Gamma(G, \mathcal{H}, X)$ decomposes as

$$p = s_1 u_1 \dots s_k u_k s_{k+1},$$

where each u_i is a geodesic segment in $\Gamma(G, \mathcal{H}, X)$ connecting the endpoints of t_i.

Define

$$M = max\{M(R, gQg^{-1}, 2\sigma) : g \in G, |g|_X \leq \sigma\},$$

where $M(R, gQg^{-1}, 2\sigma)$ is the constant given by Proposition 2.16 applied to the group G with the metric $dist_X$, the subgroups R and gQg^{-1}, and the constant 2σ.

Claim 1. For each $i \in \{1, \dots, k\}$, the endpoints of the segment u_i are at distance at most $M + 2\sigma$ from a left coset $x_i(R \cap Q_i^g)$ with respect to the metric $dist_X$. Here $x_i \in Q$ and $g_i \in G$ with $|g|_X \leq \sigma$.

To simplify notation denote the segment u_i by u. Since R is quasiconvex relative to $\mathcal{H} \cup \{Q\}$, there exists elements x and y of R such that

$$max\{dist_X(x, u_-), dist_X(y, u_+) \leq \sigma.$$

Let g denote the element $x^{-1}u_-$. Since $dist_X(x^{-1}y, gQ) \leq \sigma$ and $x^{-1}y \in R$, Proposition 2.16 implies that there is an element $z \in R \cap Q^g$ such that $dist_X(x^{-1}y, z) \leq M + \sigma$. It follows

$$max\{dist_X(u_-, x), dist_X(u_+, xz)\} \leq M + 2\sigma,$$

where $x \in R$, $|g|_X \leq \sigma$, and $z \in R \cap Q^g$.

Claim 2. For each $i \in \{1, \ldots, k\}$, every vertex of the segment u_i is at X-distance at most $(\epsilon(\lambda, c, L) + \sigma_2)$ from an element of R. This claim finish proves the lemma.

Since $x_i \in R$ and $R \cap Q^{g_i}$ is σ_2-quasiconvex relative to $\mathcal{H}$, then the left coset $x_i(R \cap Q^{g_i})$ is σ_2-quasiconvex and is a subset of R. The claim follows from a direct application of the BCP-property 2.4. $\square$

Proof of Theorem 1.1. By Corollary 3.5 every element of $\mathcal{H}_2 \setminus \mathcal{H}_1$ is hyperbolically embedded relative to $\mathcal{H}_2$. If $R < G$ is quasiconvex relative to $\mathcal{H}_2$, an induction argument using Proposition 4.1 shows that R is quasiconvex relative to $\mathcal{H}_1$.

Suppose that R is σ-quasiconvex relative to $\mathcal{H}_1$ and that $R \cap Q^g$ is quasiconvex relative to $\mathcal{H}_2$ for every $Q \in \mathcal{H}_1 \setminus \mathcal{H}_2$ and $g \in G$. Suppose that $\mathcal{H}_1 \setminus \mathcal{H}_2 = \{Q_1, \ldots, Q_l\}$. By Corollary 3.5, G is hyperbolic relative to $\mathcal{H}_2 \cup \{Q_1, \ldots, Q_{l-1}\}$. By the first part of Theorem 1.1 (already proved), $R \cap Q_l^g$ is quasiconvex relative to $\mathcal{H}_2 \cup \{Q_1, \ldots, Q_{l-1}\}$ for every $g \in G$. Since Q_l is hyperbolically embedded relative to $\mathcal{H}_2 \cup \{Q_1, \ldots, Q_{l-1}\}$, Proposition 4.2 implies that R is quasiconvex relative to $\mathcal{H}_2 \cup \{Q_1, \ldots, Q_{l-1}\}$. An argument by induction shows that R is quasiconvex relative to $\mathcal{H}_2$. $\square$

4.2 Proof of Corollary 1.3

Since Q is quasiconvex relative to $\mathcal{H}$, and P is hyperbolically embedded with respect to $\mathcal{H}$, Theorem 1.1 implies that Q is quasiconvex relative to $\mathcal{H} \cup \{P\}$. By [15, Theorem 1.1], there is a constant $C = C(P, Q, X)$ such that the homomorphism

$$Q *_{Q \cap R} R \longrightarrow G$$

is injective if $Q \cap P < R < P$ and $|g|_X \geq C$ for all $g \in R \setminus Q$.

Let R be a quasiconvex subgroup relative to $\mathcal{H}$ such that $Q \cap P < R < P$ and $|g|_X \geq C$ for all $g \in R \setminus Q$. By [15, Theorem 1.1], $\langle Q \cup R \rangle$ is quasiconvex relative to $\mathcal{H} \cup \{P\}$. We claim that $\langle Q \cup R \rangle$ is also quasiconvex relative to $\mathcal{H}$.

Let σ be the quasiconvexity constant of the subgroup Q with respect to $\mathcal{H} \cup \{P\}$. By Proposition 2.14 every infinite maximal parabolic subgroup of Q with respect to $\mathcal{H} \cup \{P\}$ is conjugate by an element of Q to a subgroup in the collection

$$\Pi = \{Q \cap H^z : H \in \mathcal{H} \cup \{P\} \text{ and } z \in G \text{ with } |z|_X \leq \sigma\}.$$

Since parabolic subgroups relative to $\mathcal{H}$ are quasiconvex relative to $\mathcal{H}$, Proposition 2.13 implies that $Q \cap H^z$ is quasiconvex relative to $\mathcal{H}$ for any $H \in \mathcal{H}$ and $z \in G$. Since P is hyperbolically embedded relative to $\mathcal{H}$, Proposition 2.13 and Theorem 2.15 imply $Q \cap P^z$ is quasiconvex relative to $\mathcal{H}$ for any $z \in G$. Therefore, every subgroup in the collection Π is quasiconvex with respect to $\mathcal{H}$.

By [15, Theorem 1.1], every infinite maximal parabolic subgroup of $\langle Q \cup R\rangle$ with respect to $\mathcal{H} \cup \{P\}$ is conjugate by an element of $\langle Q \cup R\rangle$ to maximal parabolic subgroup of Q or R, and therefore to a subgroup in the collection

$$\Pi_2 = \{Q \cap H^z : H \in \mathcal{H} \cup \{P\} \text{ and } z \in G \text{ with } |z|_X \leq \sigma\} \cup \{R\}.$$

Since R is quasiconvex relative to $\mathcal{H}$, it follows that every subgroup in the collection Π_2 is quasiconvex relative to $\mathcal{H}$.

Let $g \in G$ an consider the subgroup $K = \langle Q \cup R\rangle \cap P^g$. If K is finite then it is quasiconvex. Otherwise K is a maximal parabolic subgroup of $\langle Q \cup R\rangle$ with respect to $\mathcal{H} \cup \{P\}$. It follows that K is conjugate to a subgroup of Π_2; since conjugation preserves quasiconvexity, it follows that K is quasiconvex relative to $\mathcal{H}$. Therefore $\langle Q \cup R\rangle \cap P^g$ is quasiconvex for every $g \in G$ and, by Theorem 1.1, the subgroup $\langle Q \cup R\rangle$ is quasiconvex relative to $\mathcal{H}$.

4.3 Proof of Corollary 1.4

We follow the argument by G. Arzhantseva and A. Minasyan in [2] and argue that the subgroups $\langle f, g\rangle$ are quasiconvex relative to $\mathcal{H}$. First, we recall some results.

4.3.1 Some known results

The following proposition is contained in the proof of [2, Theorem 1.1]. For the convenience of the reader we briefly recall the argument.

Proposition 4.3. *Let G be a hyperbolic group with respect to a collection of subgroups $\mathcal{H}$. Let F be a finite subset of G. Then there exists a collection of subgroups $\mathcal{H}'$ with the following properties.*

1. *G is hyperbolic relative to $\mathcal{H}'$ and*
2. *$\mathcal{H} \subset \mathcal{H}'$,*
3. *every $H \in \mathcal{H}' \setminus \mathcal{H}$ is finite or elementary, and*
4. *every element of F is parabolic relative to $\mathcal{H}'$.*

Proof. If $f \in G$ is hyperbolic or finite order, then $\langle f\rangle$ is a finite index subgroup of a maximal elementary subgroup $H(f)$ which is hyperbolically embedded [16, Corollary 1.7 and Theorem 4.3]. The extended peripheral structure is obtained by an induction argument on the cardinality of F by adding the subgroups $H(f)$ when necessary. □

Proposition 4.4 below is a fundamental part in the proof of [2, Theorem 1.1] by Arzhantseva, Minasyan, and Osin.

Proposition 4.4. *[3][Lemma 3.8] [2, Lemma 8] Let G be a non-elementary and properly relatively hyperbolic group with respect to a collection of subgroups $\mathcal{H}$. Suppose that G has no non-trivial finite normal subgroups. Then there is a hyperbolic element h such that the subgroup $\langle h\rangle$ is hyperbolically embedded with respect to $\mathcal{H}$.*

4.3.2 Proof of Corollary 1.4

Let $\mathcal{H}'$ be the peripheral structure given by Lemma 4.3 for the set F. By Proposition 4.4 there is a hyperbolic element h of infinite order such that the subgroup $\langle h\rangle$ is hyperbolically embedded with respect to $\mathcal{H}'$.

Let f be an element of F. Since powers of hyperbolic elements are hyperbolic, the cyclic subgroups $\langle f\rangle$ and $\langle h\rangle$ intersect trivially. Applying Corollary 1.3 to the subgroups $\langle f\rangle$ and $\langle h\rangle$, there is a positive integer $n = n(f)$ such that for any $m \geq n(f)$, the subgroup $\langle f, h^m\rangle$ is isomorphic to $\langle f\rangle * \langle h^m\rangle$ and is quasiconvex relative to $\mathcal{H}'$. By Corollary 1.2 the subgroup $\langle f, h^m\rangle$ is also quasiconvex relative to $\mathcal{H}$.

To finish the proof, let $g = h^m$ where m is the minimum common multiple of $\{n(f) : f \in F\}$.

4.4 Proof of Corollary 1.6

4.4.1 Peripheral structure induced by a Quasiconvex Subgroup

We follow the exposition in [1]. Let G be a hyperbolic group and H a quasiconvex subgroup. A collection of elements $\{g_1, \dots, g_n\}$ of G is *essentially distinct* if $g_iH \neq g_jH$ for $i \neq j$. Conjugates of H by essentially distinct elements are called *essentially distinct conjugates*. The height of H is n if there exists a collection of n essentially distinct conjugates of H such that the intersection of all elements of the collection is infinite and n is maximal possible. A result by Gitik, Mitra, Rips, and Sageev states that the height of H is finite [9].

Suppose that the height of H is n. Then there are only finitely many H-conjugacy classes of intersections $H \cap H^{g_1} \cap \dots H^{g_j}$ where $j \leq n$ and $\{H, H^{g_1}, \dots, H^{g_j}\}$ are essentially distinct conjugates [1, Corollary 3.5]. Choose one subgroup of this form per H-conjugacy class to form a collection $\mathcal{E}$ of subgroups of H. The *malnormal core of H*, denoted by $\mathcal{D}$, is the collection of commensurators of elements of $\mathcal{E}$ in H.

A result by I. Kapovich and H. Short states that an infinite quasiconvex subgroup of a word hyperbolic group has a finite index in its commensurator [13]. It follows that $\mathcal{D}$ consists of quasiconvex subgroups, and

1. for any $D \in \mathcal{D}$ and $g \in H \setminus D$, the intersection $D \cap D^g$ is finite,
2. for any distinct pair $D, D' \in \mathcal{D}$ and $g \in H \setminus D$, the intersection $D^g \cap D'$ is finite.

By a theorem of Bowditch [4, Theorem 7.11], the group H is relatively hyperbolic with respect to $\mathcal{D}$.

The collection $\mathcal{D}$ gives rise to a collection of peripheral subgroups $\mathcal{P}$ for G in two steps. First change $\mathcal{D}$ to $\mathcal{D}_0$ by replacing each element of $\mathcal{D}$ by its commensurator in G. Then eliminate redundant entries of $\mathcal{D}_0$ to obtain $\mathcal{P} \subset \mathcal{D}_0$ which contains no two elements which are conjugate in G. The collection $\mathcal{P}$ is called *the peripheral structure of G induced by H*.

As the argument in the previous paragraph, all elements of $\mathcal{P}$ are commensurators of infinite quasiconvex subgroups and hence they are quasiconvex. Moreover

1. for any $P \in \mathcal{P}$ and $g \in G \setminus P$, the intersection $P \cap P^g$ is finite,
2. for any distinct pair $P, P' \in \mathcal{P}$ and $g \in G \setminus P$, the intersection $P^g \cap P'$ is finite.

By Bowditch's result [4, Theorem 7.11], the group G is hyperbolic relative to $\mathcal{P}$.

4.4.2 Proof of Corollary 1.6

That G is hyperbolic relative to $\mathcal{P}$ is explained above together with the definition of $\mathcal{P}$. Since H is a quasiconvex subgroup of G, by Theorem 1.1 (4.1) H is a quasiconvex subgroup of G relative to $\mathcal{P}$.

References

[1] I. Agol, D. Groves, and J. Manning. Residual finiteness, QCERF, and fillings of hyperbolic groups. *Geom. Topol.*, 13:1043–1073, 2009.

[2] G. Arzhantseva and A. Minasyan. Relatively hyperbolic groups are c^*-simple. *J. Funct. Anal.*, 243:345–351, 2007.

[3] G. Arzhantseva, A. Minasyan, and D. Osin. The sq-universality and residual properties of relatively hyperbolic groups. *J. of Algebra.*, 315:165–177, 2007.

[4] B.H. Bowditch. Relatively hyperbolic groups. Preprint at `http://www.warwick.ac.uk/~masgak/papers/`, 1999.

[5] F. Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963, 2003.

[6] C. Drutu and M. Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology.*, 44:959–1058, 2005. With an appendix by D. Osin and M. Sapir.

[7] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.

[8] R. Gitik. Ping-pong on negatively curved groups. *J. Algebra*, 217:65–72, 1999.

[9] R. Gitik, M. Mitra, E. Rips, and M. Sageev. Widths of subgroups. *Trans. Amer. Math. Soc.*, 350:321–329, 1998.

[10] M. Gromov. *Hyperbolic Groups*, volume 8 of *Essays in Group Theory, editor S.M. Gersten*, pages 75–263. Springer-Verlag, MSRI Series, 1987.

[11] D. Groves and J.F. Manning. Dehn filling in relatively hyperbolic groups. *Israel Journal of Mathematics*, 168:317–429, 2008.

[12] G.C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Preprint at `arXiv:0801.4596`, 2008.

[13] I. Kapovich and H. Short. Greenberg's theorem for quasiconvex subgroups of word hyperbolic groups. *Canad. J. Math.*, 48:1224–1244, 1996.

[14] J.F. Manning and E. Martínez-Pedroza. Separation of relatively quasiconvex subgroups. Preprint at `arXiv:0811.4001v2`, 2009.

[15] E. Martínez-Pedroza. Combination of quasiconvex subgroups of relatively hyperbolic groups. *Groups, Geometry and Dynamics*, 3:317–342, 2009.

[16] D. Osin. Elementary subgroups of relatively hyperbolic groups and bounded generation. *Internat. J. Algebra Comput.*, 16:99–118, 2006.

[17] D. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[18] A. Szczepański. Relatively hyperbolic groups. *Michigan Math. J.*, 45:611–618, 1998.

[19] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.